A Versatile Approach for Solving PnP, PnPf, and PnPfr Problems
–Appendix–

Gaku Nakano

Central Research Labs, NEC Corporation, Japan
g-nakano@cq.jp.nec.com

A MATLAB code for the automatic generator

```
% register the generator
setpaths;

% coefficient matrix (6x6 symmetric)
M = gbs_Matrix('M%d%d', 6, 6);
M = M - tril(M) + triu(M).';

% unknowns (1st and 2nd row of R)
syms a b c d e
r1 = [a; b; c];
r2 = [d; e; 1]; %linear constraint, R(3,3)=1

% build polynomial equations
A    = M(1:3,1:3)*r1   + M(1:3,4:6)*r2;
B    = M(1:3,4:6).'*r1 + M(4:6,4:6)*r2;
S1   = [0 -c b; c 0 -a; -b a 0]; % [r1]_x
S2   = [0 -1 e; 1 0 -d; -e d 0]; % [r2]_x
eq1  = S1*A + S2*B;              % Eq.(23)
eq2  = r2.'*A - r1.'*B;          % Eq.(26)
ceq1 = r1.'*r1 - r2.'*r2;        % Eq.(19)
ceq2 = r1.'*r2;                  % Eq.(20)
eqs  = [ceq1, ceq2, eq1(:).', eq2];

% collect known & unknown variables
unknown = {'a','b', 'c', 'd','e'};
vars    = symvar(M);
known   = {};
for var = vars, known = [known {char(var)}]; end

% call gbsolver
[res, export] = gbs_CreateCode(mfilename, eqs, known, unknown);
```

The final published version is available at https://rd.springer.com/chapter/10.1007/978-3-319-46487-9_21.
DOI: 10.1007/978-3-319-46487-9_21

B Solving the second subproblem for PnP and PnPf problems

In PnPf problem, because of $\mathbf{k} = \mathbf{0}$, we do not need to compute the third, fourth, and fifth columns of $\mathtt{L}$ in Eq. (29). Thus, $\mathtt{L}$ and $\mathbf{x}$ are represented as

$$\mathtt{L} = \begin{bmatrix} v_1 & v_1 z_1^c \\ -u_1 & -u_1 z_1^c \\ \vdots & \vdots \\ v_n & v_n z_1^c \\ -u_n & -u_n z_1^c \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} f^{-1} t_z \\ f^{-1} \end{bmatrix}. \tag{B-1}$$

The vector $\mathbf{g}$ is the same as described in Eq. (29).

In PnP problem, f^{-1} is known and $\mathbf{k} = \mathbf{0}$. From this, we obtain

$$\mathtt{L} = f^{-1} \begin{bmatrix} v_1 \\ -u_1 \\ \vdots \\ v_n \\ -u_n \end{bmatrix}, \quad \mathbf{x} = t_z, \quad \mathbf{g} = f^{-1} \begin{bmatrix} v_1 z_1^c \\ -u_1 z_1^c \\ \vdots \\ v_n z_1^c \\ -u_n z_1^c \end{bmatrix} + \begin{bmatrix} -y_1^c \\ x_1^c \\ \vdots \\ -y_n^c \\ x_n^c \end{bmatrix}. \tag{B-2}$$

C Detail formulations of KKT condition for root polishing

Let $\mathtt{I}_m$ be an $m \times m$ identity matrix and $\otimes$ be an operator representing the kronecker product. In PnPfr problem, we can write $\mathtt{C}_{(f,\mathbf{k})}$ and $\mathtt{D}_{(f,\mathbf{k})}$ as

$$\mathtt{C}_{(f,\mathbf{k})} = \mathtt{A}_1 \mathtt{W} + \mathtt{A}_2 \mathtt{Y}, \tag{C-1}$$

$$\mathtt{D}_{(f,\mathbf{k})} = \mathtt{B}_1 \mathtt{K} + \mathtt{B}_2 \mathtt{Z}, \tag{C-2}$$

where

$$\mathtt{A}_1 = \begin{bmatrix} \mathbf{0}_{n\times 3} & \begin{matrix} -\mathbf{p}_1^\top \\ \vdots \\ -\mathbf{p}_n^\top \end{matrix} & \begin{matrix} v_1\mathbf{p}_1^\top \\ \vdots \\ v_n\mathbf{p}_n^\top \end{matrix} \\ \begin{matrix} \mathbf{p}_1^\top \\ \vdots \\ \mathbf{p}_n^\top \end{matrix} & \mathbf{0}_{n\times 3} & \begin{matrix} -u_1\mathbf{p}_1^\top \\ \vdots \\ -u_n\mathbf{p}_n^\top \end{matrix} \end{bmatrix}, \quad \mathtt{A}_2 = \begin{bmatrix} \mathbf{0}_{n\times 9} & \begin{matrix} -(\mathbf{p}_1 \otimes \mathbf{d}_1)^\top \\ \vdots \\ -(\mathbf{p}_n \otimes \mathbf{d}_n)^\top \end{matrix} \\ \begin{matrix} (\mathbf{p}_1 \otimes \mathbf{d}_1)^\top \\ \vdots \\ (\mathbf{p}_n \otimes \mathbf{d}_n)^\top \end{matrix} & \mathbf{0}_{n\times 9} \end{bmatrix},$$

$$\mathtt{B}_1 = \begin{bmatrix} \mathbf{0}_{n\times 1} & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} & \begin{matrix} v_1 \\ \vdots \\ v_n \end{matrix} \\ \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} & \mathbf{0}_{n\times 1} & \begin{matrix} u_1 \\ \vdots \\ u_n \end{matrix} \end{bmatrix}, \quad \mathtt{B}_2 = \begin{bmatrix} \mathbf{0}_{n\times 3} & \begin{matrix} -\mathbf{d}_1^\top \\ \vdots \\ -\mathbf{d}_n^\top \end{matrix} \\ \begin{matrix} \mathbf{d}_1^\top \\ \vdots \\ \mathbf{d}_n^\top \end{matrix} & \mathbf{0}_{n\times 3} \end{bmatrix}, \tag{C-3}$$

$$\mathtt{W} = \begin{bmatrix} \mathtt{I}_6 & \\ & f^{-1}\mathtt{I}_3 \end{bmatrix}, \quad \mathtt{Y} = \left[\mathtt{I}_6 \otimes \mathbf{k}, \mathbf{0}_{18\times 3}\right], \quad \mathtt{Z} = \left[\mathtt{I}_2 \otimes \mathbf{k}, \mathbf{0}_{6\times 1}\right].$$

In PnPf problem, $\mathtt{Y}$ and $\mathtt{Z}$ are zero matrices due to $\mathbf{k} = \mathbf{0}$. Thus, we obtain

$$\mathtt{C}_{(f)} = \mathtt{A}_1\mathtt{W}, \tag{C-4}$$

$$\mathtt{D}_{(f)} = \mathtt{B}_1\mathtt{K}. \tag{C-5}$$

Substituting Eqs. (C-4) and (C-5) into Eqs. (30) and (31), respectively, we obtain

$$\min_{\mathtt{R},\mathbf{t},f} \; \|\mathtt{A}_1\mathtt{W}\mathbf{r} + \mathtt{B}_1\mathtt{K}\mathbf{t}\|^2 \tag{C-6}$$

and

$$\begin{aligned} \mathbf{t}_{(\mathbf{r},f)} &= -(\mathtt{K}^\top\mathtt{B}_1^\top\mathtt{B}_1\mathtt{K})^{-1}\mathtt{K}^\top\mathtt{B}_1^\top\mathtt{A}_1\mathtt{W}\mathbf{r} \\ &= -\mathtt{K}^{-1}(\mathtt{B}_1^\top\mathtt{B}_1)^{-1}\mathtt{B}_1^\top\mathtt{A}_1\mathtt{W}\mathbf{r}. \end{aligned} \tag{C-7}$$

Then, a new constrained problem for PnPf problem can be represented as

$$\begin{aligned} \min_{\mathbf{r},f} \quad & \mathbf{r}^\top\mathtt{W}\bar{\mathtt{G}}\mathtt{W}\mathbf{r} \\ \text{s.t.} \quad & \mathtt{R}^\top\mathtt{R} = \mathtt{I}, \quad det(\mathtt{R}) = 1, \end{aligned} \tag{C-8}$$

where

$$\bar{\mathtt{G}} = \mathtt{A}_1^\top\mathtt{A}_1 - \mathtt{A}_1^\top\mathtt{B}_1(\mathtt{B}_1^\top\mathtt{B}_1)^{-1}\mathtt{B}_1^\top\mathtt{A}_1. \tag{C-9}$$

We can write the KKT condition for PnPf problem without Lagrange multipliers by

$$\begin{aligned} \mathtt{R}^\top mat(\mathtt{W}\bar{\mathtt{G}}\mathtt{W}\mathbf{r}) - mat(\mathtt{W}\bar{\mathtt{G}}\mathtt{W}\mathbf{r})^\top \mathtt{R} &= \mathbf{0}_{3\times3}, \\ mat(\mathtt{W}\bar{\mathtt{G}}\mathtt{W}\mathbf{r})\mathtt{R}^\top - \mathtt{R}\ mat(\mathtt{W}\bar{\mathtt{G}}\mathtt{W}\mathbf{r})^\top &= \mathbf{0}_{3\times3}, \\ \mathtt{R}^\top\mathtt{R} - \mathtt{I} &= \mathbf{0}_{3\times3}, \\ det(\mathtt{R}) - 1 &= 0, \\ \frac{\partial}{\partial f}\mathbf{r}^\top \mathtt{W}\bar{\mathtt{G}}\mathtt{W}\mathbf{r} &= 0. \end{aligned} \tag{C-10}$$

In PnP problem, since f is known, we can calculate $\check{\mathtt{G}} = \mathtt{W}\bar{\mathtt{G}}\mathtt{W}$ and represent a new constrained problem as

$$\begin{aligned} \min_{\mathbf{r}} \quad & \mathbf{r}^\top \check{\mathtt{G}}\mathbf{r} \\ \text{s.t.} \quad & \mathtt{R}^\top\mathtt{R} = \mathtt{I}, \quad det(\mathtt{R}) = 1. \end{aligned} \tag{C-11}$$

Then, the KKT condition for PnP problem can be written by

$$\begin{aligned} \mathtt{R}^\top mat(\check{\mathtt{G}}\mathbf{r}) - mat(\check{\mathtt{G}}\mathbf{r})^\top \mathtt{R} &= \mathbf{0}_{3\times3}, \\ mat(\check{\mathtt{G}}\mathbf{r})\mathtt{R}^\top - \mathtt{R}\ mat(\check{\mathtt{G}}\mathbf{r})^\top &= \mathbf{0}_{3\times3}, \\ \mathtt{R}^\top\mathtt{R} - \mathtt{I} &= \mathbf{0}_{3\times3}, \\ det(\mathtt{R}) - 1 &= 0. \end{aligned} \tag{C-12}$$

Obviously $\bar{\mathtt{G}}$ and $\check{\mathtt{G}}$ do not contain any unknowns. Therefore we do not need to update $\bar{\mathtt{G}}$ and $\check{\mathtt{G}}$ in the Gauss-Newton iteration for PnP and PnPf problems.